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The Branch Processes of Chern-Simons (CS) p-Branes

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Abstract In this paper, by making use of *Duan*'s topological current theory, the branch process of Chern-Simons (CS) p-branes is discussed in detail. Chern-Simons (CS) p-branes are found generating or annihilating at the limit points and encountering, splitting, or merging at the bifurcation points and higher degenerated points systematically of the vector order parameter field $\vec{\phi}(x)$. Furthermore, it is also shown that CS p-branes are found splitting or merging at the degenerate point of field function $\vec{\phi}$ but the total topological charges of the CS p-branes are still unchanged.

Keywords Topological tensor current · Chern-Simons p-branes · Branch theory · High-dimensional topological defect

1 Introduction

Chern-Simons (CS) p-branes [1], which are the extended objects with p spatial dimensions in higher dimensional CS theories, play an essential role in revealing the nonperturbative structure of the modern string theories and M-theories [2–6] and so that receive much attention [7]. Just like ordinary quantum field theory can be embedded in string theory, topological Chern-Simons theory can be realized in topological open string theory [8]. CS p-branes may be considered as a straightforward generalization of the $(2+1)$-dimensional Abelian CS vortices. Over the last decade a great deal of work on the $(2+1)$-dimensional Abelian CS theory has been done [9, 10]. More recently, there has been a particular wide interest in the higher dimensional CS theories with gauge group $SO(2N+1,1)$ or $SO(2N,1)$ in $(2N+1)$ dimensions because they determine gravitational theories [11]. Higher dimensional gauge theory of CS p-branes in the phase-space is constructed in a Hamilton gauge similar to the

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ordinary W-gravity [1], and the global CS $(2N-1)$-form leads to the conservation law of a modified Nöether current [12].

In resent years, CS p-branes have also been proven to be topological defects in gauge theory [13–15]. Antisymmetric tensor gauge fields determine all of the features of a p-brane and have been widely studied in the theory of p-brane [14, 16–20]. For example, in [14, 20–24], from the perspective of a higher dimensional theory, the topological theories of p-branes in M-theory were studied. Ren et al. [6] found that the topological p-branes are created at every isolated zero of order parameter field $\vec{\phi}$. Some physicists have noticed [25, 26] that the topological defects [27, 28] are closely related to the spontaneously broken $O(M)$ symmetry group to $O(m-1)$ by m-component order parameter field $\vec{\phi}$. And $O(M)$ symmetry vector field theories are a class of models describing the critical behavior of a great variety of important physical systems [29–32], including CS p-branes. For instance, Zhao et al. [33] proposed the detailed topological theory of CS p-branes by making use of Duan's topological current theory and pointed out that the topological charges of the CS p-branes are the Winding numbers which determined by the Brouwer degrees and Hopf indices of the ϕ-mapping. The purpose of this paper is to investigate the origin and bifurcation theory of the CS p-branes, which can be regarded as a continuous study following the works done by Zhao et al.

In this paper, in light of *Duan*'s topological current theory [34, 35], a useful method, which plays an important role in studying the topological invariants [36, 37] and the topological structures [39–43, 45, 46] of CS p-branes, will study branch process [44] of CS p-branes. We display that the CS p-branes are generated from the zero points of the order parameter field $\vec{\phi}$, and their topological charges are quantized in term of the Brouwer degrees and Hopf indices of ϕ-mapping under the condition that the zero points of field $\vec{\phi}$ are regular points. While at the critical points of the order parameter field $\vec{\phi}$, i.e., the limit points and bifurcation points, there exist branch processes, the topological current of defect bifurcates and the CS p-branes split or merge at such points, which mean that CS p-branes are unstable at these points.

2 Topological Tensor Current of CS p-Branes and its Inner Topological Structure

It is well known that the d-component vector order parameter field $\vec{\phi}(x)=(\phi^1(x),\dots,\phi^d(x))$ determines the properties of the topological defect system. It can be looked upon as a smooth mapping between a D (odd)-dimensional smooth manifold X (with a metric tensor $g_{\mu\nu}$ and local coordinates $x^\mu(\mu,\nu=0,\dots,D-1)$ with $x^0=t$ being time) and an Euclidean space R^d of dimensional $d<D$ as $\phi: X\to R^d$, which gives a d-dimensional smooth vector field on X

$$\phi^a=\phi^a(x),\quad a=1,2,\dots,d. \tag{1}$$

From (1), one can express the direction unit field of the d-component vector order parameter field $\vec{\phi}(x)$ as

$$n^a=\frac{\phi^a}{\|\phi\|},\quad \|\phi\|=\sqrt{\phi^a\phi^a},\ n^an^a=1. \tag{2}$$

It is obvious in the language of differential topology that n^a is a section of the sphere bundle $S(X)$ and it can be also looked upon as a map of X onto an $(d-1)$-dimensional unit sphere S^{d-1} in order parameter space. We can see clearly that the zero points of the vector order parameter field $\vec{\phi}(x)$ are just the singular points of the unit vector $n^a(x)$, at which the direction is indefinite.

According to above discussion, in the context of the $SO(N)$ non-Abelian CS theory, by analogy with the discussion in previous work [33], one can deduce a topological tensor current for CS p-branes from the 'electric' source as follows:

$$j^{\mu_1\cdots\mu_{(D-d)}} = \frac{1}{A(S^{d-1})(d-1)!(d-2)} \frac{\varepsilon_{\mu_1\cdots\mu_{(D-d)}\mu_{(D-d+1)}\cdots\mu_D}}{\sqrt{-g_x}} \times \epsilon_{a_1a_2\cdots a_{(d-1)}a_d}\partial_{\mu_{(D-d+1)}}n^{a_1}\cdots\partial_{\mu_D}n^{a_d}. \quad (3)$$

In this expression, $A(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the surface area of $(d-1)$-dimensional unit sphere S^{d-1} and $g_x = \det(g_{\mu\nu})$ is the determinant of the metric tensor $g_{\mu\nu}$.

It is easy to see that $j^{\mu_1\cdots\mu_{(D-d)}}$ is completely antisymmetric and identically conserved, i.e.,

$$\nabla_i j^{\mu_1\cdots\mu_{(D-d)}} = 0, \quad i = 1, \ldots, D-d. \quad (4)$$

Therefore, we call the tensor $j^{\mu_1\cdots\mu_{(D-d)}}$ the $(D-d)$th-order topological tensor current.

In the following, using *Duan*'s topological current theory, the intrinsic structure of the generalized topological current $j^{\mu_1\cdots\mu_{(D-d)}}$ will be investigated. From (2), we can obtain generalized functions as the following:

$$\partial_\mu n^a = \frac{1}{\|\phi\|}\partial_\mu\phi^a + \phi^a\partial_\mu\left(\frac{1}{\|\phi\|}\right), \qquad \frac{\partial}{\partial\phi^a}\left(\frac{1}{\|\phi\|}\right) = -\frac{\phi^a}{\|\phi\|^3}. \quad (5)$$

Due to these expressions the generalized topological current (3) can be rewritten as

$$j^{\mu_1\cdots\mu_{(D-d)}} = \frac{1}{A(S^{d-1})(d-1)!(d-2)} \frac{1}{\sqrt{-g_x}} \varepsilon_{a_1\cdots a_d}\varepsilon^{\mu_1\cdots\mu_{(D-d)}\mu_{(D-d+1)}\cdots\mu_D} \times \partial_{\mu_{(D-d+1)}}\phi^a\partial_{\mu_{(D-d+2)}}\phi^{a_2}\cdots\partial_{\mu_D}\phi^{a_d}\frac{\partial}{\partial\phi^a}\frac{\partial}{\partial\phi^{a_1}}\left(\frac{1}{\|\phi\|^{d-2}}\right). \quad (6)$$

Defining a generalized Jacobian tensor

$$\varepsilon^{a_1\cdots a_d} J^{\mu_1\cdots\mu_{(D-d)}}\left(\frac{\phi}{x}\right) = \varepsilon^{\mu_1\cdots\mu_{(D-d)}\mu_{(D-d+1)}\cdots\mu_D}\partial_{\mu_{(D-d+1)}}\phi^{a_1}\partial_{\mu_{(D-d+2)}}\phi^{a_2}\cdots\partial_{\mu_D}\phi^{a_d}, \quad (7)$$

and by making use of the generalized Laplacian Green function relation in ϕ-space

$$\frac{\partial}{\partial\phi^a}\frac{\partial}{\partial\phi^a}\left(\frac{1}{\|\phi\|^{d-2}}\right) = \frac{4\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)}\delta(\phi), \quad (8)$$

we do get the δ-function structure of the topological current rigorously

$$j^{\mu_1\cdots\mu_{(D-d)}} = \left(\frac{1}{\sqrt{-g_x}}\right)\delta(\phi) J^{\mu_1\cdots\mu_{(D-d)}}\left(\frac{\phi}{x}\right). \quad (9)$$

One can find that this expression involves the total defect information of the system and indicates that all the defects are located at the zero points of the order parameter field $\vec{\phi}(x)$, i.e., $j^{\mu_1\cdots\mu_{(D-d)}} \neq 0$ only when $\phi = 0$, which is just the singularity of $j^{\mu_1\cdots\mu_{(D-d)}}$. In other words, the topological tensor current does not vanish only at the zero points of the vector field. In detail, the kernel of the ϕ-mapping is the singularity of the topological tensor current $j^{\mu_1\cdots\mu_{(D-d)}}$ in the odd-dimensional smooth manifold X, i.e., the zeros of ϕ-mapping labels the inner structure of the topological tensor current. This is the essential of *Duan*'s topological current theory.

Suppose that the vector field $\vec{\phi}(x)$ possesses l isolated zeros, according to the implicit function theorem, when the zeros are regular points of ϕ-mapping, i.e., the rank of the Jacobian matrix $[\partial_\mu \phi^a]$ is d, the solutions of $\vec{\phi}(x) = 0$ can be parameterized by

$$x^\mu = z_i^\mu(u^1, u^2, \ldots, u^{(D-d)}), \quad i = 1, \ldots, l, \tag{10}$$

where the subscript i represents the ith solution and the parameters $\mu^I (I = 1, \ldots, D-d)$ span a $(D-d)$-dimensional submanifold, with the metric tensor $g_{IJ} = g_{\mu\nu}(\frac{\partial x^\mu}{\partial u^I})(\frac{\partial x^\nu}{\partial u^J})$ denoted by N_i, which corresponds to CS p-brane $(p = D-d-1)$ with spatial p-dimension and $(D-d)$-dimensional singular submanifold N_i is its world volume. When u^1 and $\vec{u} = (u^2, \ldots, u^{(D-d)})$ are taken to be the time-like evolution parameter and space-like position parameter, respectively, the inner structure of $j^{\mu_1\cdots\mu_{(D-d)}}$ represents l $(D-d-1)$-dimensional topological defects moving in X.

Clearly, the tensor current $j^{\mu_1\cdots\mu_{(D-d)}}$ does not vanish only on the world volume manifolds N_i $(i = 1, \ldots, l)$, each of which corresponds to an 'electric' CS p-brane. Therefore, sometimes, the CS p-brane may be considered as topological defects, and the vector field $\vec{\phi}^a(x)$ $(a = 1, \ldots, d)$ may be looked upon as the generalized order parameter for CS p-branes.

As we have proven in [47], for all the $(D-d)$-dimensional singular submanifolds N_i, a common d-dimensional normal submanifold M_i exists in X with $D-d+d = D$ and M_i is spanned by the parameters v^A with the parameter equation

$$x^\mu = x^\mu(v^1, \ldots, v^d), \quad \mu = 1, \ldots, D, \tag{11}$$

and the metric tensor $g_{AB} = g_{\mu\nu}(\frac{\partial x^\mu}{\partial v^A})(\frac{\partial x^\nu}{\partial v^B})$ $(A, B = 1, \ldots, d)$. The intersection points of M_i and N_i are denoted by p_i $(i = 1, \ldots, l)$ which can be expressed parametrically by $v^A = p_i^A$. In differential topology, it is said that M_i is transversal to N_i at p_i, i.e.

$$T_{p_i}(X) = T_{p_i}(M_i) + T_{p_i}(N_i), \tag{12}$$

where $T_{p_i}(X)$, $T_{p_i}(M_i)$ and $T_{p_i}(N_i)$ are the tangent space of X, M_i and N_i at p_i, respectively.

In the following, we will investigated the inner topological structure of CS p-branes. As pointed out in [48], the generalized winding number W_i of n^a at p_i can be defined by the Gauss map n: $\partial M_i \to S^{d-1}$

$$W_i = \frac{1}{A(S^{d-1})(d-1)!} \int_{\partial M_i} n^*(\varepsilon_{a_1\cdots a_d} n^{a_1} dn^{a_2} \Lambda \cdots \Lambda dn^{a_d}), \tag{13}$$

where ∂M_i is the boundary of the neighborhood M_i of p_i with $p_i \notin \partial M_i$, $\partial M_i \cap \partial M_j = \emptyset$ and n^* is the pull-back of n. The generalized winding number is a topological invariant and also called the degree of Gauss map. It means that, when the point v^a covers ∂M_i once, the unit vector n^a will cover a region $n[\partial M_i]$, whose area is $A(S^{d-1})$, W_i times, i.e., the unit vector v^a will cover the unit sphere S^{d-1} W_i times. Using the Stokes' theorem in exterior differential form and duplicating the above process, we obtain the compact form of W_i,

$$W_i = \int_{M_i} \delta(\phi(v)) J\left(\frac{\phi}{v}\right) d^d v. \tag{14}$$

According to the δ-function theory, the δ-function $\delta(\phi)$ can be expanded by

$$\delta(\phi) = \sum_{i=1}^{l} c_i \delta(N_i), \tag{15}$$

where the coefficients c_i must be positive, i.e., $c_i = |c_i|$. $\delta(N_i)$ is the δ-function in space-time X on a submanifold N_i,

$$\delta(N_i) = \int_{N_i} \frac{1}{\sqrt{-g_x}} \delta^n\left(\vec{x} - \vec{z}_i(u^1, \ldots, u^{(D-d)})\right) \sqrt{-g_u} d^{D-d} u, \tag{16}$$

where $g_u = \det(g_{IJ})$. Substituting (15) into (14), and calculating the integral, with positivity of c_i adopted, the coefficient c_i becomes

$$c_i = \frac{\beta_i \sqrt{-g_v}}{|J(\frac{\phi}{v})_{p_i}|} = \frac{\beta_i \eta_i \sqrt{-g_v}}{J(\frac{\phi}{v})_{p_i}}, \tag{17}$$

where $\beta_i = |W_i|$ is a positive integer called the Hopf index of *Duan*'s topological current theory on M_i and $\eta_i = \mathrm{sgn} J(\frac{\phi}{v})_{p_i} = \pm 1$ is the Brouwer degree. Substituting the expression of c_i and (15) into (9), we gain the total expansion of the rank-$(D-d)$ topological current,

$$j^{\mu_1 \cdots \mu_{(D-d)}} = \frac{1}{\sqrt{-g_x}} \sum_{i=1}^{l} \frac{\beta_i \eta_i \sqrt{-g_v}}{J(\frac{\phi}{v})|_{p_i}} \delta(N_i) J^{\mu_1 \cdots \mu_{(D-d)}}\left(\frac{\phi}{x}\right). \tag{18}$$

From the above equation, we conclude that the total expansion of $\delta(x)$, which includes the topological information about β_i and η_i, labels the inner structure of $j^{\mu_1 \cdots \mu_{(D-d)}}$. In (18), it just represents l CS p-branes with topological charges $g_i = W_i = \beta_i \eta_i$, which is just Winding number, moving in the D-dimensional spacetime manifold X. The $(D-d)$-dimensional singular submanifolds $N_i (i = 1, \ldots, l)$ are their world sheets in the space-time.

3 The Branch Processes of CS *p*-Branes

With the discussion mentioned above, the results in the above section are obtained straightly from the topological viewpoint under the condition $J(\frac{\phi}{v}) \neq 0$, i.e., at the regular points of the order parameter field $\vec{\phi}$. However, when the condition fails, the usual implicit function theorem [49] is of no use. The above discussion will change in some way and lead to the branch process.

In this section, we will discuss the branch processes of CS p-branes. In order to simplify our study, we select u^1 as the time-like evolution parameter t, and let the space-like parameter $u^I = \sigma^I$ $(I = 2, \ldots, D-d)$ to be fixed, i.e., just select one point of CS p-branes to study. In this case, the Jacobian matrices $j^{\mu_1\cdots\mu_{(D-d)}}(\frac{\phi}{y})$ are reduced to

$$J^{AI_1\cdots I_{D-d-1}}\left(\frac{\phi}{y}\right)\equiv J^A\left(\frac{\phi}{y}\right),\qquad J^{ABI_1\cdots I_{D-d-2}}\left(\frac{\phi}{y}\right)=0,$$

$$J^{(d+1)\cdots D}\left(\frac{\phi}{y}\right)\equiv J\left(\frac{\phi}{v}\right)=0,\quad A,B=1,\ldots,(d+1),\ I_j=d+2,\ldots,D, \tag{19}$$

for $y^A=v^A$ $(A\le d)$, $y^{d+1}=t$, $y^{d+I}=\sigma^I$ $(I\ge 2)$. These branch points are determined by the $d+1$ equation,

$$\phi^a(v^1,\ldots,v^d,t,\vec{\sigma})=0,\quad a=1,\ldots,d, \tag{20}$$

and

$$\phi^{d+1}(v^1,\ldots,v^d,t,\vec{\sigma})\equiv J\left(\frac{\phi}{v}\right)=0. \tag{21}$$

Now, we denote one of the zero points as (t^*,p_i). Let us explore what happens to CS p-branes. In *Duan*'s topological current theory, there are usually two kinds of branch points, the limit points and bifurcation points [50], satisfying

$$J^1\left(\frac{\phi}{y}\right)\bigg|_{(t^*,p_i)}\neq 0, \tag{22}$$

and

$$J^1\left(\frac{\phi}{y}\right)\bigg|_{(t^*,p_i)}=0, \tag{23}$$

respectively.

3.1 Branch Process at the Limit Point

If the Jacobian

$$J^1\left(\frac{\phi}{y}\right)\bigg|_{(t^*,P_i)}\neq 0, \tag{24}$$

we can use the Jacobian $J^1(\frac{\phi}{x})$ instead of $J(\frac{\phi}{x})$ for the purpose of using the implicit function theorem. This means we will replace the time-like variable t by v^1. Then we have a unique solution of (20) in the neighborhood of the limit point (t^*,p_i)

$$t=t(v^1,\vec{\sigma}),\qquad v^i=v^i(v^1,\vec{\sigma}),\quad i=2,3,\ldots,d, \tag{25}$$

with $t^*=t(p_i^1,\vec{\sigma})$. We call the critical point (t^*,p_i) the limit point. In the present case, we know that

$$\frac{dv^1}{dt}\bigg|_{(t^*,p_i)}=\frac{J^1(\frac{\phi}{y})}{J(\frac{\phi}{v})}\bigg|_{(t^*,p_i)}=\infty, \tag{26}$$

i.e.,

$$\frac{dt}{dv^1}\bigg|_{(t^*,p_i)}=0. \tag{27}$$

Then the Taylor expansion of $t = t(v^1)$ at the limit point (t^*, p_i) is

$$t - t^* = \frac{1}{2} \left. \frac{d^2 t}{(dv^1)^2} \right|_{(t^*, p_i)} (v^1 - p_i^1)^2, \tag{28}$$

which is a parabola in $v^1 - t$ plane. From (28) we can obtain two solutions $v_1^1(t, \vec{\sigma})$ and $v_2^1(t, \vec{\sigma})$, which give two branch solutions (world lines of CS p-branes). If

$$\left. \frac{d^2 t}{(dv^1)^2} \right|_{(t^*, p_i)} > 0. \tag{29}$$

We have the branch solutions for $t > t^*$; otherwise, we have the branch solutions for $t < t^*$. These two cases are related to the origin and annihilation of CS p-branes.

3.2 Branch Process at the Bifurcation Point

In this section we have the restrictions of (23) at the bifurcation points (t^*, p_i),

$$\left. J\left(\frac{\phi}{v}\right) \right|_{(t^*, p_i)} = 0, \qquad \left. J^1\left(\frac{\phi}{y}\right) \right|_{(t^*, p_i)} = 0, \tag{30}$$

which lead to an important fact that the function relationship between t and v^1 is not unique in the neighborhood of the bifurcation point (t^*, p_i). In our dynamic form of current, we have

$$\left. \frac{dv^1}{dt} \right|_{(t^*, p_i)} = \left. \frac{J^1(\frac{\phi}{y})}{J(\frac{\phi}{v})} \right|_{(t^*, p_i)}, \tag{31}$$

which under (30) directly shows that the direction of the integral curve of (31) is indefinite at (t^*, p_i). That is why the very point (t^*, p_i) is called a bifurcation point.

Assume that the bifurcation point (t^*, p_i) has been found from (30). We know that, at the bifurcation point, the rank of the Jacobian matrix $[\frac{\partial \phi}{\partial v}]$ is less than d. In order to derive the calculating method, we consider the rank of the Jacobian matrix $[\frac{\partial \phi}{\partial v}]$ is $d - 1$. Suppose

$$J_1\left(\frac{\phi}{v}\right) = \begin{vmatrix} \phi_2^1 & \phi_3^1 & \cdots & \phi_d^1 \\ \phi_2^2 & \phi_3^2 & \cdots & \phi_d^2 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_2^{d-1} & \phi_3^{d-1} & \cdots & \phi_d^{d-1} \end{vmatrix} \tag{32}$$

and its determinant $\det J(\frac{\phi}{v})$ does not vanish at the point (t^*, p_i), where ϕ_A^a stands for $(\frac{\partial \phi^a}{\partial v^A})$ $(a = 1, \ldots, d-1;\ A = 2, \ldots, d)$. The implicit function theorem implies that there exists one and only one system of function relationships

$$v^A = f^A(v^1, t, \sigma^2, \ldots, \sigma^k), \quad A = 2, 3, \ldots, D, \tag{33}$$

with the partial derivatives

$$f_1^A = \frac{\partial v^A}{\partial v^1}, \qquad f_t^A = \frac{\partial v^A}{\partial t}, \quad A = 2, 3, \ldots, D. \tag{34}$$

Then, for $a = 1, \dots, d-1$ we have

$$\phi^a = \phi^a\big(v^1, f^2(v^1, t, \vec{\sigma}), \dots, f^d(v^1, t, \vec{\sigma}), t, \vec{\sigma}\big) \equiv 0, \tag{35}$$

which gives

$$\sum_{A=2}^{d} \frac{\partial \phi^a}{\partial v^A} f_1^A = -\frac{\partial \phi^a}{\partial v^1}, \quad a = 1, \dots, d-1, \tag{36}$$

$$\sum_{A=2}^{d} \frac{\partial \phi^a}{\partial v^A} f_t^A = -\frac{\partial \phi^a}{\partial t}, \quad a = 1, \dots, d-1, \tag{37}$$

from which we can calculate the first order derivatives of f^A: f_1^A and f_t^A. Denoting the second order partial derivatives as

$$f_{11}^A = \frac{\partial^2 v^A}{(\partial v^1)^2}, \qquad f_{1t}^A = \frac{\partial^2 v^A}{\partial v^1 \partial t}, \qquad f_{tt}^A = \frac{\partial^2 v^A}{(\partial t)^2}, \tag{38}$$

and differentiating (36) with respect to v^1 and t, respectively, we obtain

$$\sum_{A=2}^{d} \phi_A^a f_{11}^A = -\sum_{A=2}^{d} \left[2\phi_{A1}^a f_1^A + \sum_{B=2}^{d} (\phi_{AB}^a f_1^B) f_1^A \right] - \phi_{11}^a, \quad a = 1, 2, \dots, d-1, \tag{39}$$

$$\sum_{A=2}^{d} \phi_A^a f_{1t}^A = -\sum_{A=2}^{d} \left[\phi_{At}^a f_1^A + \phi_{A1}^a f_t^A + \sum_{B=2}^{d} (\phi_{AB}^a f_t^B) f_1^A \right] - \phi_{1t}^a, \quad a = 1, 2, \dots, d-1, \tag{40}$$

$$\sum_{A=2}^{d} \phi_A^a f_{tt}^A = -\sum_{A=2}^{d} \left[2\phi_{At}^a f_t^A + \sum_{B=2}^{d} (\phi_{AB}^a f_t^B) f_t^A \right] - \phi_{tt}^a, \quad a = 1, 2, \dots, d-1, \tag{41}$$

where

$$\phi_{AB}^a = \frac{\partial^2 \phi^a}{\partial v^A \partial v^B}, \qquad \phi_{At}^a = \frac{\partial^2 \phi^a}{\partial v^A \partial t}. \tag{42}$$

The differentiation of (36) with respect to v^1 gives the same expression as (40). If we use the Gaussian elimination method to the three vectors at the right-hand sides of the formulas (39), (40), and (41), we can get the three partial derivatives f_{11}^A, f_{1t}^A, and f_{tt}^A. Notice that (39), (40), and (41) have the same coefficient matrix $J_1(\frac{\phi}{v})$, which are assumed to be nonzero, and we should substitute the values of the partial derivatives f_1^A and f_t^A into the right-hand sides of the three equations.

Here we must note that the above discussions do not relate to the last component $\phi^d(v^1, \dots, v^d, t, \vec{\sigma})$. With the aim of finding the different directions of all branch curves at the bifurcation point, let us investigate the Taylor expansion of

$$F(t, v^1, \vec{\sigma}) = \phi^d\big(t, v^1, f^2(t, v^1, \vec{\sigma}), \dots, f^d(t, v^1, \vec{\sigma}), \vec{\sigma}\big), \tag{43}$$

which must vanish at the bifurcation point, i.e.,

$$F(t^*, p_i) = 0. \tag{44}$$

From (43), the first-order partial derivatives of $F(t, v^1, \vec{\sigma})$ is

$$\frac{\partial F}{\partial t} = \phi^d_t + \sum_{A=2}^{d} \phi^d_A f^A_t, \qquad \frac{\partial F}{\partial v^1} = \phi^d_1 + \sum_{A=2}^{d} \phi^d_A f^A_1. \tag{45}$$

On the other hand, making use of (36), (37), (45), and Cramer's rule [38], it is not difficult to prove that the first equation of (30) can be rewritten as

$$J\left(\frac{\phi}{v}\right)\bigg|_{(t^*, p_i)} = \frac{\partial F}{\partial v^1} \det J_1\left(\frac{\phi}{v}\right)\bigg|_{(t^*, p_i)} = 0. \tag{46}$$

Since

$$\det J_1\left(\frac{\phi}{v}\right)\bigg|_{(t^*, p_i)} \neq 0 \tag{47}$$

which is our assumption, the above equation gives

$$\frac{\partial F}{\partial v^1}\bigg|_{(t^*, p_i)} = 0. \tag{48}$$

With the same reasons, we can prove that

$$\frac{\partial F}{\partial t}\bigg|_{(t^*, p_i)} = 0. \tag{49}$$

The second-order partial derivatives of the function $F(t, v^1, \vec{\sigma})$ are easily found to be

$$\frac{\partial^2 F}{(\partial v^1)^2} = \phi^d_{11} + \sum_{A=2}^{d}\left[2\phi^d_{A1} f^A_1 + \phi^d_A f^A_{11} + \sum_{B=2}^{d}(\phi^d_{AB} f^B_1) f^A_1\right], \tag{50}$$

$$\frac{\partial^2 F}{\partial t \partial v^1} = \phi^d_{1t} + \sum_{A=2}^{d}\left[\phi^d_{1A} f^A_t + \phi^d_{tA} f^A_1 + \phi^d_A f^A_{t1} + \sum_{B=2}^{d}(\phi^d_{AB} f^B_t) f^A_1\right], \tag{51}$$

$$\frac{\partial^2 F}{\partial t^2} = \phi^3_{tt} + \sum_{A=2}^{d}\left[2\phi^d_{At} f^A_t + \phi^m_A f^A_{tt} + \sum_{B=2}^{d}(\phi^d_{AB} f^B_t) f^A_t\right], \tag{52}$$

which at (t^*, p_i) are denoted by

$$A = \frac{\partial^2 F}{(\partial v^1)^2}\bigg|_{(t^*, p_i)}, \qquad B = \frac{\partial^2 F}{\partial t \partial v^1}\bigg|_{(t^*, p_i)}, \qquad C = \frac{\partial^2 F}{(\partial t)^2}\bigg|_{(t^*, p_i)}. \tag{53}$$

According to the *Duan*'s topological current theory, the Taylor expansion of the solution of $F(t, v^1, \vec{\sigma})$ in the neighborhood of the bifurcation point can generally be denoted as

$$A(v^1 - p^1_i)^2 + 2B(v^1 - p^1_i)(t - t^*) + C(t - t^*)^2 = 0, \tag{54}$$

which leads to

$$A\left(\frac{dv^1}{dt}\right)^2+2B\left(\frac{dv^1}{dt}\right)+C=0 \tag{55}$$

and

$$C\left(\frac{dt}{dv^1}\right)^2+2B\left(\frac{dt}{dv^1}\right)+A=0, \tag{56}$$

where A, B, and C are three constants. The solutions of (55) or (56) give different directions of the branch curves at the bifurcation point. There are four possible cases, which show the physical meanings of the bifurcation points.

Case 1 ($A\neq 0$) For $\Delta=4(B^2-AC)>0$ from (55) we get two different directions of CS p-branes

$$\left.\frac{dv^1}{dt}\right|_{(t^*,p_i)}=\frac{-B\pm\sqrt{B^2-AC}}{A}. \tag{57}$$

It is the intersection of two CS p-branes with different directions at the bifurcation point, which means that two CS p-branes meet and then depart from each other at the bifurcation point.

Case 2 ($A\neq 0$) For $\Delta=4(B^2-AC)=0$ from (55) we obtain only one direction of CS p-branes

$$\left.\frac{dv^1}{dt}\right|_{(t^*,p_i)}=\frac{-B}{A} \tag{58}$$

which includes three important situations: (a) One CS p-brane splits into two CS p-branes at the bifurcation point. (b) Two CS p-branes merge into one CS p-brane at the bifurcation point. (c) Two CS p-branes tangentially encounter at the bifurcation point.

Case 3 ($A=0, C\neq 0$) For $\Delta=4(B^2-AC)=0$, we have

$$\left.\frac{dt}{dv^1}\right|_{1,2}=\frac{-B\pm\sqrt{B^2-AC}}{C}=\begin{cases}0,\\ -\frac{2B}{C}.\end{cases} \tag{59}$$

There are two important cases: (a) Three CS p-branes merge into one CS p-brane at the bifurcation point. (b) One CS p-brane resolves into three CS p-branes at the bifurcation point.

Case 4 ($A=C=0$) Equations (55) and (56) give respectively

$$\frac{dv^1}{dt}=0,\qquad \frac{dt}{dv^1}=0. \tag{60}$$

This case is similar to Case 3.

3.3 The Bifurcation of CS p-Branes at a Higher Degenerate Point

In the preceding section we studied the case that the rank of the Jacobian matrix $[\frac{\partial\phi}{\partial v}]$ of (21) is $d-1$. In this section, we investigate the case that the rank of the Jacobian matrix is $d-2$ (for the case that the rank of the Jacobian matrix $[\frac{\partial\phi}{\partial v}]$ is lower than $d-2$, the discussion is the same way). Let the $(d-2)\times(d-2)$ submatrix $J_2(\frac{\phi}{v})$ of the Jacobian matrix $[\frac{\partial\phi}{\partial v}]$ be

$$J_2\left(\frac{\phi}{v}\right)=\begin{vmatrix} \phi_3^1 & \phi_4^1 & \cdots & \phi_d^1 \\ \phi_3^2 & \phi_4^2 & \cdots & \phi_d^2 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_3^{d-2} & \phi_4^{d-2} & \cdots & \phi_d^{d-2} \end{vmatrix} \tag{61}$$

and suppose that $\det J_2(\frac{\phi}{v})|_{(t^*,p_i)}\neq 0$. With the same seasons of obtaining (33), we can have the function relations

$$v^A=f^A(v^1,v^2,t,\vec{\sigma}),\qquad A=3,4,\ldots,d. \tag{62}$$

In order to determine the values of the first and second order partial derivatives with respect to t, v^1 and v^2, one can substitute the relationship (62) into the last two equations of the system (20). Then, we get

$$\begin{aligned} F_1(t,v^1,v^2,\vec{\sigma})&=\phi^{d-1}\big(t,v^1,v^2,f^3(t,v^1,v^2,\vec{\sigma}),\ldots,f^d(t,v^1,v^2,\vec{\sigma}),\vec{\sigma}\big)=0,\\ F_2(t,v^1,v^2,\vec{\sigma})&=\phi^{d}\big(t,v^1,v^2,f^3(t,v^1,v^2,\vec{\sigma}),\ldots,f^d(t,v^1,v^2,\vec{\sigma}),\vec{\sigma}\big)=0. \end{aligned} \tag{63}$$

For calculating the partial derivatives of the function F_1 and F_2 with respect to t, v^1 and v^2, one can take notice of (62) and use six similar expressions to (48), i.e.,

$$\left.\frac{\partial F_j}{\partial t}\right|_{(t^*,p_i)}=0,\qquad \left.\frac{\partial F_j}{\partial v^1}\right|_{(t^*,p_i)}=0,\qquad \left.\frac{\partial F_j}{\partial v^2}\right|_{(t^*,p_i)}=0,\qquad j=1,2. \tag{64}$$

So the Taylor expansions of F_1 and F_2 can be written in the neighborhood of $(t^*,\,p_i)$ by

$$\begin{aligned} F_j(t,v^1,v^2,\vec{\sigma})\approx{}& A_{j1}(v^1-p_i^1)^2+A_{j2}(v^1-p_i^1)(v^2-p_i^2)\\ &+A_{j3}(t-t^*)(v^1-p_i^1)+A_{j4}(v^2-p_i^2)^2+A_{j5}(v^2-p_i^2)(t-t^*)\\ &+A_{j6}(t-t^*)^2=0, \end{aligned} \tag{65}$$

where $j=1,2$.

Dividing (65) by $(t-t^*)^2$ and taking the limit $t\to t^*$, one obtains the two quadratic equations of $\frac{dv^1}{dt}$ and $\frac{dv^2}{dt}$,

$$A_{j1}\left(\frac{dv^1}{dt}\right)^2+A_{j2}\frac{dv^1}{dt}\frac{dv^2}{dt}+A_{j3}\frac{dv^1}{dt}+A_{j4}\left(\frac{dv^2}{dt}\right)^2+A_{j5}\frac{dv^2}{dt}+A_{j6}=0, \tag{66}$$

and further, eliminating the variable $\frac{dv^1}{dt}$, one has the equation of $\frac{dv^2}{dt}$ in the form of a determinant

$$\begin{vmatrix} A_{1a} & A_{12}Q + A_{23} & A_{14}Q^2 + A_{15}Q + A_{16} & 0 \\ 0 & A_{1a} & A_{12}Q + A_{23} & A_{14}Q^2 + A_{15}Q + A_{16} \\ A_{21} & A_{22}Q + A_{23} & A_{24}Q^2 + A_{25}Q + A_{26} & 0 \\ 0 & A_{21} & A_{22}Q + A_{23} & A_{24}Q^2 + A_{25}Q + A_{26} \end{vmatrix} = 0, \tag{67}$$

with the variable $v = \frac{dv^2}{dt}$, which is a four-order equation of $\frac{dv^2}{dt}$

$$a_1\left(\frac{dv^2}{dt}\right)^4 + a_2\left(\frac{dv^2}{dt}\right)^3 + a_3\left(\frac{dv^2}{dt}\right)^2 + a_4\left(\frac{dv^2}{dt}\right) + a_5 = 0. \tag{68}$$

Hence, different directions of the branch curves at the second-order degenerate point is structured. The largest number of different branch curves is four. If the degree of the degeneracy of the matrix $[\frac{\partial\phi}{\partial v}]$ is higher, i.e., the rank of the matrix $[\frac{\partial\phi}{\partial v}]$ is lower than the present $(d-2)$ case, then the procedure of deduction will be more complicated. In general supposing the rank of the matrix $[\frac{\partial\phi}{\partial v}]$ is $(d-s)$, the number of possible different directions of the branch curves is 2^s at most.

Furthermore, since the topological current is a conserved current, the sum of the topological charges of these final CS p-branes must be equal to that of the original CS p-branes at the bifurcation point [38], i.e.,

$$\sum_i \beta_{l_i}\eta_{l_i} = \sum_f \beta_{l_f}\eta_{l_f} \tag{69}$$

for fixed l. Furthermore, from the above studies, we see that the generation, annihilation, and bifurcation of CS p-branes are not gradually changed, but suddenly changed at the critical points.

4 Conclusion

Our conclusions can be summarized as follows: First, we give a review of the topological quantization of CS p-branes and point out that the topological charges are the Winding numbers which determined by the Brouwer degrees of the ϕ-mapping. Second, we investigate the branch process of CS p-branes at the limit points, bifurcation points and higher degenerated points systematically by virtue of *Duan*'s topological current theory and the implicit function theorem. Third, we see that the generation, annihilation, and bifurcation of CS p-branes are not gradually changed, but suddenly changed at the critical points. Finally, we would like to point out that all the results in this paper have been obtained only from the viewpoint of topology. All of above derivations are not referred to any special solutions of CS p-brane (the exact analytical expression of vector field $\vec{\phi}$ which supports the existence of CS p-branes.), therefore our results are general in the theory of CS p-brane and can be used to any solutions of CS p-branes.

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